Available online at www.sciencedirect.com

ScienceDirect

Procedia Materials Science 3 (2014) 750 – 755

Procedia Materials Science

www.elsevier.com/locate/procedia

20th European Conference on Fracture (ECF20)

# On the exactness of truncated crack-tip stress expansions

Gaëtan Hello [a,] *, Mabrouk Ben Tahar[b]

[a]Université d'Evry Val d'Essonne - UFRST/LMEE, 40 rue de Pelvoux, 91020 Courcouronnes, France
[b]Université de Technologie de Compiègne - Laboratoire Roberval UMR7337, Centre de Recherches de Royallieu, 60203 Compiègne, France

**Abstract**

Williams series appear to be the most favored analytical tool for the description of mechanical fields near crack-tips in planar domains. For practical use, these series are generally truncated. A common belief is to consider that the more terms are kept, the more accurate the representation will be. Based on closed-form series expressions, this belief is shown to be only partially true. Asymptotic expansions converge within series convergence disks as expected, but truncated series can also provide exact values for the stress field. This property can be easily observed with the map of relative error comparing truncated series solutions for stress with complex exact ones. The series remainder appears to be equal to zero on curves emanating from the crack-tip. Their number and their initial angles are shown to be related to the zeros of a Williams eigen-function that depends on the number of terms kept in the truncated series.


Selection and peer-review under responsibility of the Norwegian University of Science and Technology (NTNU), Department of Structural Engineering

*Keywords:* Crack-tip stress, Williams series, Higher order terms.

## 1. Introduction

The stress field at the vicinity of a crack-tip in a plane medium may be described using the so-called Williams series, Williams (1952):

* Corresponding author. Tel.: +33 169477548; fax: +33 169477599.
*E-mail address:* gaetan.hello@ufrst.univ-evry.fr


Selection and peer-review under responsibility of the Norwegian University of Science and Technology (NTNU), Department of Structural Engineering
doi:10.1016/j.mspro.2014.06.123

$$\sigma_{ij}(r,\theta) = \sum_{m=1}^{2}\sum_{k=-\infty}^{\infty} a_k^m f_k^{m,ij}(\theta)\, r^{\frac{k}{2}-1} \tag{1}$$

with index $m$ associated to the fracture mode; $a_k^m$ coefficients related to the geometric configuration, load and mode;
$f_k^{m,ij}(\theta)$ angular functions depending on stress component and mode.

Each term of the series is the product of three factors. Two of them concern respectively the angular and radial dependencies of the field. Their general expressions are known analytically and are the same for all fracture configurations, Owen and Fawkes (1983); Karihaloo and Xiao (2001). All the specific information related to the actual problem (geometry and loading conditions) is held by the third factor. Hence, for each fracture problem, there exists a specific infinite set of multi-order stress intensity factors. Concerning the determination of these sets, research has been mainly focused on the term of index $k = 1$ (the stress intensity factor $K_i$ associated to the stress singularity $r^{-1/2}$, Tada et al. (2000)) and on the next one with $k = 2$ (the T-stress associated to a constant stress state $r^0$, Smith et al. (2001)). Only a few papers have proposed analytical expressions for higher order stress intensity factors. For instance, the first terms are implicitly given for some different configurations in Paris (2002); Tada and Paris (2005) while whole sets of coefficients are given for some specific examples in Theocaris and Spyropoulos (1983); Yan and Yang (1993); Hello et al. (2012, 2013).
The authors have proposed closed-form asymptotic expansions for different fracture configurations. Expressions are provided for the multi-order stress intensity factors in either mode I or mode II problems using both power series (Williams series) and Laurent series. Thanks to these expressions, the convergence behavior of crack-tip stress expansions may be studied. The existence of the expected radii of convergence is observed. Rates of convergence are quantified. With the description of the crack-tip stress field by series arises the problem of truncation influence. The accuracy of the series representation is generally thought to improve as the number of terms increases. However, this common belief appears to be only partially true. The asymptotic expansions converge within the convergence disk as expected. However, truncated series can provide exact results for stress at some specific locations in the plane. In this communication, numerical investigations are performed in order to illustrate this intriguing property of Williams series. An analytical explanation is then proposed in order to predict some of the features of the loci of exactness for truncated series.

## 2. Analytical background

### *2.1. Complex solutions*

If the exactness of truncated Williams series is to be assessed, exact solutions are then required to play the role of reference. The use of complex analysis has proven very efficient in dealing with planar linear elasticity. In the first half of the 20th century, many problems have been solved analytically using complex analytic potentials, Muskhelishvili (1953); England (1971). In the case of fracture configurations, the popular Westergaard approach, Westergaard (1939); Sih (1966); Sanford (1979); Tada et al. (2000); Sanford (2003) provides convenient expressions for instance for mode-I stress state:

$$\sigma_{11}^1(z) = 2\,\mathrm{Re}\left[\phi_1{}'(z)\right] - 2x_2\ \mathrm{Im}\left[\phi_1{}''(z)\right] + C_1 \tag{2}$$

$$\sigma_{22}^1(z) = 2\,\mathrm{Re}\left[\phi_1{}'(z)\right] + 2x_2\ \mathrm{Im}\left[\phi_1{}''(z)\right] - C_1 \tag{3}$$

$$\sigma_{12}^1(z) = -2x_2\ \mathrm{Re}\left[\phi_1{}''(z)\right] \tag{4}$$

Determination of the complex potential $\phi_1'$ and constant C1 is performed for each problem with the satisfaction of boundary conditions on the crack and at infinity. The solution of the mode-I problem depicted in Fig.1 is for example given by Muskhelishvili (1953):

$$\phi_1'(z) = \frac{\sigma_{22}^{\infty}}{2} \cdot \frac{z}{\left(z^2 - a^2\right)^{\frac{1}{2}}} + (\alpha - 1)\frac{\sigma_{22}^{\infty}}{4}, \; C_1 = (\alpha - 1)\frac{\sigma_{22}^{\infty}}{2} \tag{5}$$

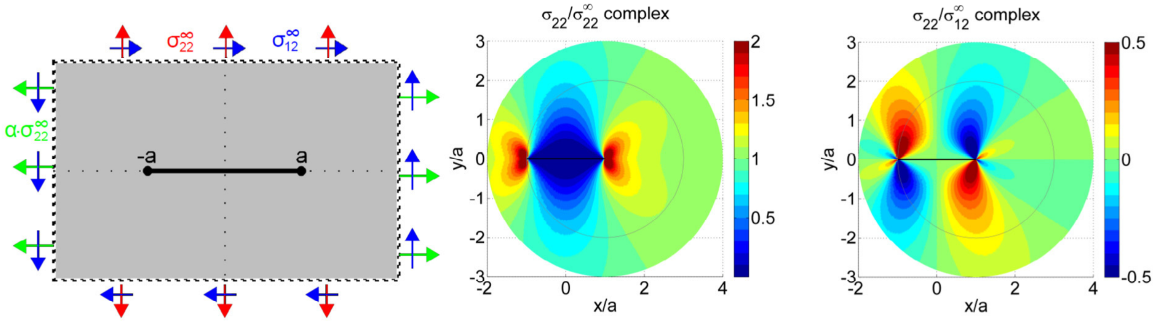


Fig. 1. (left) Fracture configuration of an infinite cracked domain submitted to remote loads in mode-I and mode-II, (center) mode-I 22 normalized complex stress field for pure mode-I load, (right) mode-II 22 normalized complex stress field for pure mode-II load.

*2.2. Series solutions*

Starting with Williams initial work, Williams (1952), the representation of mechanical fields at the vicinity of crack-tips is now generally performed with series. However, closed-form complete expressions for these series have rarely been given in the literature even though radial and angular functions in Eq. 1 are well known, Owen and Fawkes (1983); Karihaloo and Xiao (2001), for mode-I angular functions in particular:

$$f_k^{1,11}(\theta) = \frac{k}{2}\left[(2 + k/2 + (-1)^k)\cos(k/2 - 1)\theta - (k/2 - 1)\cos(k/2 - 3)\theta\right] \tag{6}$$

$$f_k^{1,22}(\theta) = \frac{k}{2}\left[(2 - k/2 - (-1)^k)\cos(k/2 - 1)\theta + (k/2 - 1)\cos(k/2 - 3)\theta\right] \tag{7}$$

$$f_k^{1,12}(\theta) = \frac{k}{2}\left[(k/2 - 1)\sin(k/2 - 3)\theta - (k/2 + (-1)^k)\sin(k/2 - 1)\theta\right] \tag{8}$$

In the specific case of the fracture configuration of Fig.1, the full set of mode-I higher order stress intensity factors has been determined, Hello et al. (2012):

$$a_{2n+1}^1 = \frac{(-1)^{n+1}(2n)!}{2^{3n+\frac{1}{2}}(n!)^2(2n-1)} \frac{\sigma_{22}^{\infty}}{a^{n-\frac{1}{2}}}, \; n \geq 0$$

$$a_2^1 = \frac{\sigma_{22}^{\infty}(\alpha - 1)}{4} \tag{12}$$

$$a_k^1 = 0, \text{ otherwise}$$

And the expression of stress component $\sigma_{22}^1$ is given by:

$$\sigma_{22}^1(r,\theta) = \sum_{n=0}^{\infty} a_{2n+1}^1 f_{2n+1}^{1,22}(\theta)\, r^{n-\frac{1}{2}} \tag{13}$$

## 3. Study of truncated series

### *3.1. Numerical investigation*

It is possible to describe $\sigma^1_{22}$ stress state for the problem depicted in Fig. 1 with either the complex approach Eqs. 3,5 or the series approach Eqs. 7,12,13. Results obtained with truncated series (7, 9, 11, 13 terms in Eq. 1, equivalent to 4, 5, 6, 7 terms in Eq. 13) are then compared with exact complex ones. In Fig. 2, the successive series fields for $\sigma^1_{22}$ are presented. In particular, the maps of relative errors are plotted (angular discretisation performed with 2001 points). Truncated series exhibit the astonishing property of being able to provide exact results along curves emanating from the crack-tip. The number of curves increases as the number of terms kept in the truncated series progresses: 2, 4, 6, 8 curves for 7, 9, 11, 13 terms.

### *3.2. Analytical investigation*

The radius of convergence R of the series representation for $\sigma^1_{22}$ can be expressed with d'Alembert's ratio test:

$$R = \lim_{n\to+\infty} R_n \; (= 2a \text{ here}) \text{ with } R_n = \frac{a^1_{2n+1} f^{1,22}_{2n+1}(\theta)}{a^1_{2n+3} f^{1,22}_{2n+3}(\theta)} \tag{14}$$

The convergence of $R_n$ is not necessarily uniform. Indeed, even though the final radius is finite, an "intermediate" radius of convergence can be infinite when the denominator of $R_n$ is equal to zero. For the example considered, it happens when $f^{1,22}_{2n+3}(\theta) = 0$. Hence, the zeros of Williams angular functions provide meaningful information about the potential exactness of truncated series. In Fig. 3, relative errors for $\sigma^1_{22}$ are plotted for different truncated series (k = 7, 9, 11, 13 terms) on 3 circles r/a = 0.5, 1, 1.5. It appears that the number of angles leading to 0 relative error is equal to k-5. This number is even and angles are symmetric. For a given number of terms, the angles are not the same for the different radii but appear to converge as $r \to 0$. Following the previous comment regarding infinite "intermediate" radius of convergence, angular functions $f^{1,22}_k(\theta)$, k = 9, 11, 13, 15 are plotted in Fig. 3 as well. These functions possess exactly k-7 symmetric zeros within $]-\pi, \pi[$. The zeros of $f^{1,22}_{2n+3}(\theta)$ appear to correspond to the zeros of truncated series with 2n + 1 terms when $r \to 0$ as expected.

## 4. Conclusion

Truncated Williams series have been shown to be able to provide exact results. Exactness can be achieved in the plane on curves emanating from the crack-tip. Depending on the number of terms held by the truncated series considered, a specific Williams angular function defines the number of these curves and their initial angle.

## References

England, A.H., 1971. Complex variable method in elasticity. Dover Publications, New York.

Hello, G., Ben Tahar, M., Roelandt, J.M., 2012. Analytical determination of coefficients in crack-tip stress expansions for a finite crack in an infinite plane medium. International Journal of Solids and Structures 49, 556–566.

Hello, G., Ben Tahar, M., Roelandt, J.M., 2013. Improvement of crack-tip stress series with pad approximants, in: 13th International Conference on Fracture. Beijing, China, June 16-21.

Karihaloo, B.L., Xiao, Q.Z., 2001. Accurate determination of the coefficients of elastic crack tip asymptotic field by a hybrid crack element with p-adaptivity. Eng. Fract. Mech. 68, 1609 – 1630.

Muskhelishvili, N., 1953. Some Basic Problems of Mathematical Theory of Elasticity. Noordhoff, Leyden.

Owen, D.R.J., Fawkes, A.J., 1983. Engineering fracture mechanics: numerical methods and applications. Pineridge Press, Swansea.

Paris, P.C., 2002. Secondary elastic crack tip stresses which may influence very slow fatigue crack growth. Fatigue Fract. Eng. Mater. Struct. 25, 791–794.

Sanford, R., 1979. A critical re-examination of the westergaard method for solving opening-mode crack problems. Mechanics Research Communications 6, 289 – 294.

Sanford, R., 2003. Principles of Fracture Mechanics. Prentice Hall.

Sih, G.C., 1966. On the Westergaard method of crack analysis. Int. J. Fract. 2, 628 – 631.

Smith, D.J., Ayatollahi, M.R., Pavier, M.J., 2001. The role of t-stress in brittle fracture for linear elastic materials under mixed-mode loading. Fatigue & Fracture of Engineering Materials & Structures 24, 137–150.

Tada, H., Paris, P.C., 2005. Secondary elastic crack tip stresses which may influence very slow fatigue crack growth - additional results. Int. J. Fatigue 27, 1307 – 1313.

Tada, H., Paris, P.C., Irwin, G.R., 2000. The stress analysis of cracks handbook. ASME, New York.

Theocaris, P., Spyropoulos, C., 1983. Photoelastic determination of complex stress intensity factors for slant cracks under biaxial loading with higher-order term effects. Acta Mechanica 48, 57–70.

Westergaard, H., 1939. Bearing pressures and cracks. Journal of Applied Mechanics A, 49–53.

Williams, M.L., 1952. Stress singularities resulting from various boundary conditions in angular corners of plates in extension. J. Appl. Mech. 19, 526–528.

Yan, X., Yang, X., 1993. On a numerical investigation of boundary collocation procedure and williams series convergency, in: Int. Conf. Frac. 8.

Fig. 2. (left) Normalized series solutions with 7,9,11,13 terms, (right) Log10 of relative errors with respect to complex exact solutions.

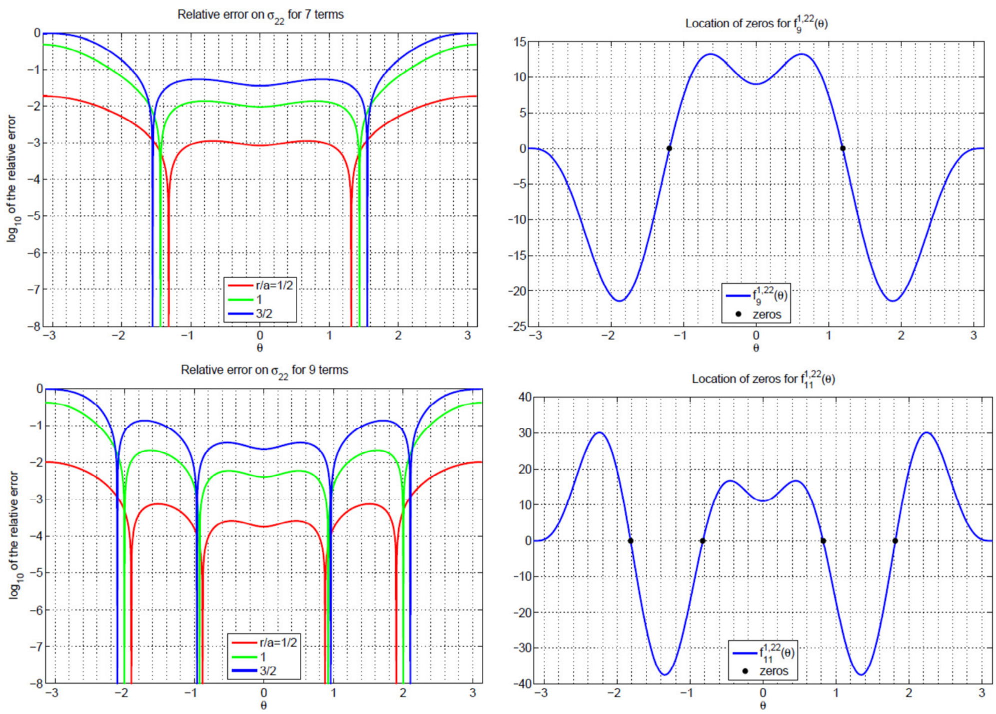


Fig. 3. (left) Relative errors of truncated series solutions with 7, 9, 11, 13 terms on circles r/a = 0.5, 1, 1.5, (right) Williams angular functions for k = 9, 11, 13, 15.